Stability of a Charged Particle in a Combined Penning-Ioffe Trap

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The axial symmetry of a familiar Penning trap is broken by adding the radial magnetic field of an Ioffe trap. Despite the resulting loss of a confinement theorem, stable orbits related to adiabatic invariants are identified, expressions are given for their frequencies, and resonances that must be avoided are characterized. It seems feasible to experimentally realize the new Penning-Ioffe trap to test these theoretical predictions. It also may be possible to simultaneously confine cold positrons and antiprotons in a Penning-Ioffe trap, along with any cold antihydrogen they may form.

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The stability of charged particles in a Penning trap (a uniform magnetic field along the symmetry axis of an electrostatic quadrupole) makes it one of the versatile devices of modern physics. Classical and quantum descriptions are elegant and exact [1]. Individual elementary particles in a Penning trap are used to test QED, determine the fine structure constant, test *CPT* with leptons and baryons, and carry out accurate mass spectroscopy. Many electrons and ions stored in such traps allow measurements of the masses of unstable nuclei, ion cyclotron resonance analysis of pharmaceuticals and biological materials, and (often in closely related Malmberg traps [2]) permit studies of single component plasmas.

The stability is closely related to axial symmetry; the resulting conservation of angular momentum gives rise to a confinement theorem [3]. Neither one charged particle nor a dense single-component plasma can spread perpendicularly to the magnetic field and leave a Penning or Malmberg trap. In this Letter we investigate the breakdown of this symmetry and confinement theorem. We add the radial magnetic quadrupole field of an Ioffe trap — an experimentally realizable modification, characterized by only one parameter. (An alternative is distorting the electrostatics of the trap [4].) A Ioffe trap is a familiar way to confine neutral particles [5]. An intriguing question is whether a Penning-Ioffe trap (e.g., Fig. 1a) could confine cold antihydrogen atoms along with the charged antiprotons ($\bar{p}$) and positrons (e^+) from which they form. The stability of the charges is crucial; they must remain confined long enough for neutral atoms to form.

The Penning-Ioffe system and the analysis of the motion of a charged particle within it are simple and clean, yet nontrivial. Despite the breakdown of axial symmetry we find stable orbits that are associated with adiabatic invariants and have simple geometrical representations. Resonant instabilities arise but can be avoided. A guiding-center approximation [6], a perturbation expansion using the method of multiple time scales [7], and exact numerical calculations are compared and discussed.

A charged particle (charge q and mass m) in a magnetic field, $B_0\hat{z}$, orbits in a right-handed circle about $\hat{z}$ at a positive cyclotron frequency $\omega_c = -qB_0/m$, with the right choice of direction for $\hat{z}$. This field and an electrostatic quadrupole (Fig. 2a) form a Penning trap. A charge at $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ acquires a potential energy,

$$W = m\omega_z^2[z^2 - (x^2 + y^2)/2]/2, \qquad (1)$$

with a strength ω_z that will be identified as an angular frequency. The ratio $\epsilon \equiv \omega_z/\omega_c$ indicates the relative strength of the electric and magnetic binding. The trap is stable when $\epsilon < 1/\sqrt{2}$. Typically $\epsilon \ll 1$, as in the first simultaneous confinement of cold $\bar{p}$ (with $\epsilon = 1.5 \times 10^{-2}$) and e^+ (with $\epsilon = 2 \times 10^{-4}$) [8].

Adding the radial magnetic field of an Ioffe trap to $B_0\hat{z}$,

$$\vec{B} = B_0[\hat{z} + (x\hat{x} - y\hat{y})/R_0], \qquad (2)$$

introduces a distance scale, R_0. Axial symmetry, present for large R_0, is destroyed as R_0 is reduced (e.g., by increasing the Ioffe current). Superconducting coils could produce an Ioffe gradient, $C_1 = B_0/R_0$, as large as 40 T/m, even for a bias field $B_0 = 2$ T, whereupon $R_0 = 5$ cm. A trapped particle is typically centered so that $r \ll R_0$.

An experimental realization (Fig. 1a) could direct the magnetic field of a solenoid (not pictured) along the axis of the stacked rings of an open-access Penning trap [9]. The Ioffe field would come from currents through vertical Ioffe bars and through "pinch coils" above and below. The

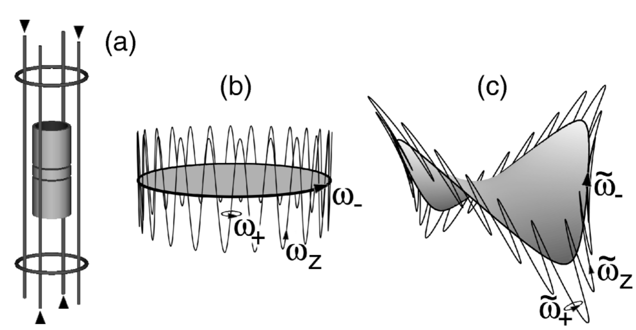

FIG. 1. (a) Open access Penning trap electrodes, with vertical current bars and pinch coils of an Ioffe trap. Orbits for a charged particle in a Penning trap (b) without and (c) with a radial Ioffe field.

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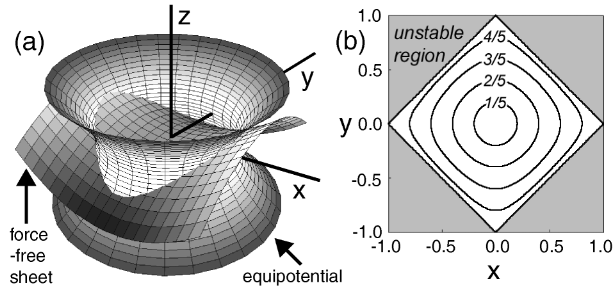

FIG. 2. (a) The force-free sheet and an equipotential of the electrostatic quadrupole. (b) Projections of stable magnetron orbits upon the xy plane lie within a square.

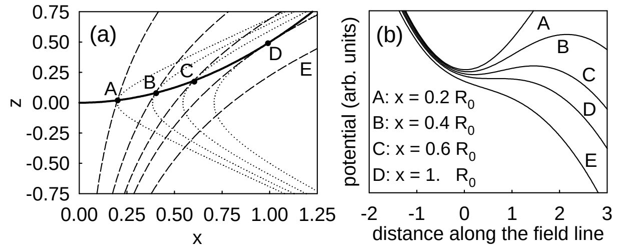

FIG. 3. (a) Magnetron orbits (dots) lie at the intersection of the force-free sheet (solid line) and equipotentials of the electrostatic quadrupole (dotted line), with magnetic field lines (dashed line). (b) The effective axial well depth decreases with increasing magnetron radius.

latter can be away from the central region where charged particles are trapped, so only the leading term of the radial magnetic field from Ioffe bars is in Eq. (2).

To simplify the equations of motion, from now on we scale times by $(\omega_c)^{-1}$ and distances by R_0, yielding

$$\ddot{x} = -\dot{y} + \epsilon^2 x/2 - y\dot{z}\,, \tag{3}$$

$$\ddot{y} = \dot{x} + \epsilon^2 y/2 - x\dot{z}\,, \tag{4}$$

$$\ddot{z} = -\epsilon^2 z + x\dot{y} + y\dot{x}\,. \tag{5}$$

The dynamics of a charge in a Penning-Ioffe trap (as in a Penning trap) are characterized by the single parameter ϵ; in useful traps $\epsilon \ll 1$. The nonlinear terms (e.g., $y\dot{z}$) are smaller than the linear terms so long as $r \ll 1$.

The nonlinear terms vanish near the center of the trap as well as when the Ioffe field is removed. The familiar solutions [1] then describe the three uncoupled oscillations of a charged particle in an ideal Penning trap.

$$u \equiv x + iy = ae^{i\omega_- t} + be^{i\omega_+ t}, \tag{6}$$

$$z = \mathrm{Re}(ce^{i\omega_z t})\,, \tag{7}$$

$$\omega_\pm = [1 \pm \sqrt{1 - 2\epsilon^2}\,]/2\,. \tag{8}$$

The oscillation frequencies are of different orders in the small parameter ϵ. The circular cyclotron oscillation is at high frequency $\omega_+ \approx 1$. The orthogonal harmonic axial oscillation is at intermediate frequency $\omega_z = \epsilon$, and the circular magnetron motion is at low frequency $\omega_- \approx \frac{1}{2}\epsilon^2$.

This frequency hierarchy is maintained in the Penning-Ioffe trap for small ϵ, the limit we consider first. Adiabatic invariants approximately preserve the exact Penning separation into three motions (Fig. 1c).

The fast cyclotron motion is perpendicular to the local magnetic field, rather than to $\hat{z}$, at the local cyclotron frequency $\tilde{\omega}_+ \approx \sqrt{1 + x^2 + y^2}$. This motion is much faster than any other and can be described by its magnetic moment, $M \approx mv_+^2/(2|\vec{B}|)$, located at a "guiding center" [6]. This moment is an adiabatic invariant [10]: as the local magnetic field changes, the cyclotron velocity v_+ (and radius) adjusts to keep M constant. Damping of the cyclotron motion (by synchrotron radiation for electrons, and by coupling to a tuned circuit for protons and ions) typically keeps it in thermal equilibrium with blackbody radiation or a cold resistor at some low temperature T. The resulting moment, with energy $MB \approx kT$, is typically much too small to noticeably affect the orbits.

The intermediate frequency "axial" oscillation is along the local magnetic field direction, as in the Penning trap. However, this direction is not $\hat{z}$, and the oscillation is not centered on the xy plane. Figure 3a shows magnetic field lines, $z = \ln(x/x_0)$, that cross the xy plane at x_0. The guiding center experiences an electric force parallel to its field line, $F_\parallel = -\vec{\nabla} W \cdot \hat{B}$. In the stable region of the trap, $F_\parallel$ is a restoring force that vanishes on

$$z = (x^2 - y^2)/2\,. \tag{9}$$

This force-free sheet (Fig. 2a) is the center of the axial oscillation; the guiding center damps to this surface when a resonant tuned circuit removes its axial energy. Figure 3b shows corresponding potential wells that are deeper nearer the center of the trap. Small oscillations in these are at the local axial oscillation frequency

$$\tilde{\omega}_z = \omega_z\sqrt{1 - x^2 - y^2}/\sqrt{1 + x^2 + y^2}\,. \tag{10}$$

The axial motion has an adiabatic invariant $J \approx E_z/\tilde{\omega}_z$. As the magnetron motion (discussed next) slowly changes a particle's radial position and hence its axial frequency, its axial energy E_z adjusts with $\tilde{\omega}_z$ to keep J constant.

The magnetron orbit is essentially the intersection of the force-free sheet and an equipotential of the electrostatic quadrupole potential (Fig. 2a). The guiding center is on the equipotential because the magnetron kinetic energy is much smaller than the potential energy, as in the Penning trap. Equations (3)–(5), with time scaled by ω_-^{-1}, can be integrated exactly for small ϵ to obtain an integral (easy to evaluate numerically) for the magnetron frequency,

$$\frac{1}{\tilde{\omega}_-} = \frac{8}{\pi\epsilon^2}\int_0^{\frac{\sqrt{2x_0^2 - x_0^4}}{2}} \frac{1 - 2v^2/\sqrt{1 - 2x_0^2 + x_0^4 + 4v^2}}{\sqrt{1 + v^2 - \sqrt{1 - 2x_0^2 + x_0^4 + 4v^2}}}\,dv\,, \tag{11}$$

for an orbit crossing the xz plane at $x = x_0$. Magnetron orbits near the center of the trap are circular with angular frequency ω_-. Larger orbits include odd harmonics of $\tilde{\omega}_-$ as they develop "corners"; symmetry under rotations about $\hat{z}$ by π suppresses even harmonics. The flux Φ enclosed by the magnetron orbit is an adiabatic invariant. For static magnetic fields, the invariance of Φ is equivalent to conservation of energy and adds nothing. If the Ioffe field strength was increased slowly (e.g., just before cold antihydrogen is formed), the magnetron radius and frequency would adjust to keep Φ constant.

The stable motions of a charge in a Penning-Ioffe trap pertain within an identifiable region. These orbits should be stable for exponentially long times [10] provided that resonances are avoided and adiabatic invariants are not otherwise broken. We discuss stability for small ϵ.

Projections of stable magnetron orbits on the xy plane lie within a square stability boundary (Fig. 2b) for small axial energy. Axial energy shrinks the boundary. Stable magnetron orbits of increasing radius (A–D in Fig. 3a) differ in effective axial well depth (Fig. 3b). This depth vanishes at D, the orbit whose projection is the square boundary. Beyond D there are no stable intersections between the force-free sheet and an energy equipotential.

Resonances $\tilde{\omega}_z = 2\tilde{\omega}_-$ must be avoided since these can cause the breakdown of the adiabatic invariants. The difficulty arises because the Ioffe field couples the axial and magnetron motions. A magnetron orbit takes the particle up and down in the z direction through two cycles (Fig. 1c), effectively "driving" the axial motion at angular frequency $2\tilde{\omega}_-$. If the axial frequency coincides with this "drive" frequency for a particular orbit, magnetron energy is transferred to the axial oscillation. Energy removed from the magnetron motion makes its radius grow until the particle eventually leaves the trap.

There are secondary resonances due to magnetron orbits away from the trap center having Fourier components at odd harmonics of $\tilde{\omega}_-$. These resonances are at $\tilde{\omega}_z = 2N\tilde{\omega}_-$, where $N > 1$ is an odd integer. We find that the frequencies for these shift out of resonance quickly enough with increasing radius to eliminate resonance and precipitous growth in the radial orbit size.

The resonances can be located using expansions around a Penning trap orbit [Eqs. (6) and (7)], utilizing the method of multiple time scales [7]. The expansions also reveal general features of the orbits, verify the guiding-center approximation, provide insight into the adiabatic invariants, and facilitate future experimental studies. We start from a Penning trap orbit with real a, and with b and c both real and small. We expand in powers of a, avoiding artificial divergences (sometimes called "secular resonances") by allowing the expansion amplitudes and phases to be relatively slowly varying functions of time.

We define $f(\omega) \equiv \omega^2 + \omega + \epsilon^2$, $g(\omega) \equiv \epsilon^2 - \omega^2$, and $h(\omega) \equiv 1 - 2\omega$ to display the Penning-Ioffe frequencies:

$$\tilde{\omega}_- = \omega_- + \frac{\omega_-^2}{g(2\omega_-)h(\omega_-)}a^2 + \frac{\omega_-^3}{g^2(2\omega_-)h(\omega_-)}\left[\frac{2}{h(\omega_-)} + \frac{\omega_-}{h^2(\omega_-)} - \frac{3\omega_-}{f(3\omega_-)} + \frac{8\omega_-^2}{g(2\omega_-)h(\omega_-)}\right]a^4 + \ldots, \tag{12}$$

$$\tilde{\omega}_z = \omega_z + \frac{\omega_z}{4}\left[\frac{1}{f(\omega_- - \omega_z)} + \frac{1}{f(\omega_- + \omega_z)}\right]a^2 + \frac{\omega_z}{2}\left[1 + \frac{7\epsilon^2}{2} + \ldots\right]a^4 - \frac{\omega_z}{4}\left[1 + \frac{15\epsilon^2}{4} + \ldots\right]a^4\cos 4\tilde{\omega}_- t + \ldots, \tag{13}$$

$$\tilde{\omega}_+ = \omega_+ + \frac{(\omega_+ + \omega_-)^2}{2g(\omega_+ + \omega_-)h(\omega_+)}a^2 - \frac{1}{8}[1 + 13\epsilon^2 + \ldots]a^4 + \frac{1}{8}\left[1 + \frac{7\epsilon^2}{2} + \ldots\right]a^4\cos 4\tilde{\omega}_m t + \ldots. \tag{14}$$

The exact a^4 coefficients for $\tilde{\omega}_z$ and $\tilde{\omega}_+$ are complicated enough that we display only the first terms in an expansion in ϵ. Frequency modulations arise from magnetron motion through the spatially varying magnetic field.

Figure 4a gives the magnetron orbit sizes at which the unwanted resonances $\tilde{\omega}_z = 2\tilde{\omega}_-$ occur for small cyclotron and axial energies. The series expansion (solid curve) is valid for orbits that are not too large. The guiding-center values (dashed curve) pertain for larger orbits and small ϵ. For a weak electrostatic field (i.e., small ϵ) the lowest order resonances are near the stability square in Fig. 2b. For a stronger electrostatic field (i.e., larger ϵ) the resonances occur for smaller orbits. The dots are example resonances identified using Fourier transforms of numerical solutions of the equation of motion. To the right of this resonance boundary, the adiabatic invariants which separate the axial and magnetron motion break down. We find no stable solutions in this region, except in the Penning trap limit of very small x. To the left of the resonance boundary, however, a charged particle in a Penning-Ioffe trap seems to be stable.

To check the stability, we use a Runge-Kutta algorithm to integrate the equations of motion [Eqs. (3)–(5)] for long times. In one example (Fig. 4a) we use $B_0 = 2$ T, $R_0 = 13$ cm, $\epsilon = 0.05$, and begin with a magnetron radius of

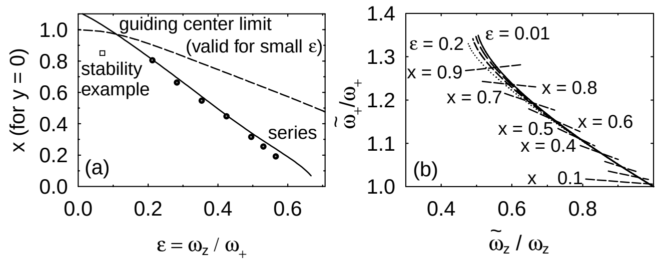

FIG. 4. (a) Magnetron orbit size that is troubled by resonance at $\tilde{\omega}_z = 2\omega_-$. Points are numerical confirmations. (b) Predicted relationship of the axial and the magnetron frequencies could be tested experimentally and used to determine the magnetron radius at $y = 0$.

0.85 and negligible axial and cyclotron energy. We detect no radial growth during 4×10^6 magnetron orbits; this is 31 s of real time.

The orbits are Fourier series in harmonics of the eigenfrequencies. For pure magnetron motion and small ϵ,

$$u_- = ae^{i\tilde{\omega}_- t} + \left[\frac{a^3}{8} + \frac{3a^5}{64}\right]e^{-3i\tilde{\omega}_- t} + \frac{a^5}{64}e^{5i\tilde{\omega}_- t}, \tag{15}$$

$$z_- = \left[\frac{a^2}{2} + \frac{a^4}{8} + \frac{3a^6}{64}\right]e^{2i\tilde{\omega}_- t} + \frac{3a^6}{128}e^{6i\tilde{\omega}_- t} \tag{16}$$

to $O(a^6)$. Substitution into Eqs. (1) and (9) explicitly confirms that this orbit lies on the electrostatic quadrupole and the force-free sheet. The exact coefficients of a^n for $\epsilon \neq 0$ are complicated functions of ϵ.

A small cyclotron oscillation adds

$$u_+ = be^{i\tilde{\omega}_+ t} - \frac{a^2 b}{4}e^{-i(\tilde{\omega}_+ + 2\tilde{\omega}_-)t} + \frac{a^4 b}{8}\left[e^{-i(\tilde{\omega}_+ + 2\tilde{\omega}_-)t} - \frac{1}{2}e^{-i(\tilde{\omega}_+ - 2\tilde{\omega}_-)t}\right], \tag{17}$$

$$z_+ = -abe^{i(\tilde{\omega}_+ + \tilde{\omega}_-)t} + \frac{a^3 b}{4}\left[e^{i(\tilde{\omega}_+ + \tilde{\omega}_-)t} - \frac{1}{2}e^{i(\tilde{\omega}_+ - 3\tilde{\omega}_-)t}\right], \tag{18}$$

$$b = b(t) = b(0)e^{-\frac{ia^4}{8}\cos 4\tilde{\omega}_- t}. \tag{19}$$

A small axial oscillation adds

$$u_z = \frac{ac}{2}[e^{-i(\tilde{\omega}_- - \tilde{\omega}_z)t} + e^{-i(\tilde{\omega}_- + \tilde{\omega}_z)t}] + \frac{a^3 c}{16}[e^{i(3\tilde{\omega}_- - \tilde{\omega}_z)t} + e^{i(3\tilde{\omega}_- + \tilde{\omega}_z)t}], \tag{20}$$

$$z_z = ce^{i\tilde{\omega}_z t}. \tag{21}$$

Substitution in the leading terms for M, J, and Φ also verifies that these are invariants through order a^4 for small ϵ. There is a delicate exact cancellation of the frequency and amplitude modulation terms.

Our predictions could be tested experimentally, starting with the predicted stability of a charge in a Penning-Ioffe trap. Second, the relationship of $\tilde{\omega}_z$ and $\tilde{\omega}_+$ could be compared with Eqs. (13) and (14) (Fig. 4b)—also a way to measure a. Third, the predicted resonant instability at $\tilde{\omega}_z = 2\tilde{\omega}_-$ could be confirmed as a function of a. Fourth, the frequency modulation spectra of $\tilde{\omega}_z$ and $\tilde{\omega}_+$ [Eqs. (13) and (14)] could be investigated.

In summary, a radial Ioffe field provides an experimentally realizable way to break the axial symmetry and a resulting confinement theorem for a Penning trap. Adiabatic invariants lead to the surprising prediction that a charged particle in a Penning-Ioffe trap is nonetheless stable for orbits within a stable volume, if resonances are avoided. The orbits have simple geometrical representations, expressions for the oscillation frequencies are given, and experimentally testable predictions are discussed.

Confining antiprotons and positrons in a nested version [11] of a Penning-Ioffe trap, along with cold antihydrogen atoms that are formed, now seems feasible for low particle densities. At a higher density, yet to be determined, collisions could break the adiabatic invariants, space charge could modify resonance frequencies, and collective plasma modes could be crucial. These effects may be more pronounced in a Malmberg-Ioffe trap [12] where oscillation frequencies are less well defined. The relationship of density and stability for charged plasmas in a Penning-Ioffe trap remains to be investigated.

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